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AN ENERGETIC FORMULATION FOR THE LINEAR VISCOELASTIC PROBLEM. PART I: THEORETICAL RESULTS AND FIRST CALCULATIONS

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SUMMARY

An extremal principle is formulated for the linear viscoelastic problem with general viscous kernel. This is an extension of the classical total potential energy principle of the linear elasticity. Then a discretized formulation in space and time is shown for frame structures, using finite element technique. Several numerical examples, for two different kinds of viscoelastic materials, testify the accuracy and reliability of the proposed method. The matrix conditioning indexes obtained are compared with those achieved by applying the least square method.

KEY WORDS Viscoelasticity Frame structures Variational methods

1. INTRODUCTION

The idea of substituting a given problem with one equivalent in variational form is certainly not new. The interest for such a formulation is justified by the power of the so-called 'direct methods' of variational calculus. They are both worth a 'qualitative' study of the associated problems (existence and uniqueness of the solution, regularity, etc.) and a 'quantitative' numerical study (like for the development of finite element techniques). In addition, from a numerical point of view, the value of the functional may be used as a measure of the convergence during the integration and the evaluation of the error of the approximate solution.

Along this path of reasoning, since 1950s, a number of attempts have been made to reformulate the viscoelastic problem in variational terms. Main contributions in the field exist, for example, those by Biot,[1] Onat,[2] Gurtin,[3] Brilla,[4] Christensen,[5] Rafalski,[6–8] Reiss and Haug,[9] Huet,[10] leading to variational formulations of the viscoelastic problem for hereditary type materials or under particular conditions, but not for general kernels.

In the present paper reference is made to the case of a non-homogeneous, non-isotropic viscoelastic solid with general viscous kernel. After explaining the linear viscoelastic problem (Section 2), an equivalent extremal formulation is shown (Section 3). The presented extremal formulation derives, as a particular case, from the family of extremal principles proposed in[11] on the basis of Tonti's general theory of variational formulation of non-linear problems.[12] Later, by using the finite element technique in space and the Ritz method in time, a discretized formulation, limited to frame structures, is presented in Section 4.

Several numerical examples are also shown (Section 5). These testify the accuracy and reliability of the method. The work is concluded with a critical discussion of numerical results by referring to matrix conditioning indexes of linear system coefficients obtained by discretization (Section 6).

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2. THE LINEAR VISCOELASTIC PROBLEM FORMULATION

2.1. The linear viscoelastic constitutive law

Let $\sigma_{ij}(\mathbf{x};t)$, $\varepsilon_{ij}(\mathbf{x};t)$, $R_{ijhk}(\mathbf{x};t,\tau)$ be respectively the stress tensor, the strain tensor and the relaxation kernel. A representation in direct form of the linear viscoelastic constitutive law is the following (supposing $\sigma_{ij}(\mathbf{x};\,t) = 0$ and $\varepsilon_{ij}(\mathbf{x};\,t) = 0$ for $t < t_0$):

$$\sigma_{ij}(\mathbf{x};\,t) = R_{ijhk}(\mathbf{x};\,t,\,t_0)\varepsilon_{hk}(\mathbf{x};\,t_0) + \int_{t_0^+}^{t} R_{ijhk}(\mathbf{x};\,t,\,\tau)\,\mathrm{d}\varepsilon_{hk}(\mathbf{x};\,\tau) \tag{1}$$

where t is the time variable ($t_0 \leqslant t < +\infty$) and τ is the integration variable. The constitutive law in inverse form is

$$\varepsilon_{ij}(\mathbf{x};\,t) = \Phi_{ijhk}(\mathbf{x};\,t,\,t_0)\sigma_{hk}(\mathbf{x};\,t_0) + \int_{t_0^+}^{t} \Phi_{ijhk}(\mathbf{x};\,t,\,\tau)\,\mathrm{d}\sigma_{hk}(\mathbf{x};\,\tau) \tag{2}$$

where $\Phi_{ijhk}(\mathbf{x};\,t,\,\tau)$ is the creep kernel. In (1) and (2) the time integrals are used in Stieltjes's way.* It is further supposed that the relaxation and creep kernels satisfy the following symmetries:

$$R_{ijhk}(\mathbf{x};\,t,\tau) = R_{hkij}(\mathbf{x};\,t,\tau) \tag{3}$$

$$\Phi_{ijhk}(\mathbf{x};\,t,\,\tau) = \Phi_{hkij}(\mathbf{x};\,t,\,\tau) \tag{4}$$

For isotropic materials the viscous kernels depend on two time functions only. In this case one obtains the following relaxation kernel:

$$R_{ijhk}(\mathbf{x};\,t,\tau) = \tfrac{1}{3}[R_2(\mathbf{x};\,t,\tau) - R_1(\mathbf{x};\,t,\tau)]\delta_{ij}\delta_{hk} + \tfrac{1}{2}R_1(\mathbf{x};\,t,\tau)[\delta_{ih}\delta_{jk} + \delta_{ik}\delta_{jh}] \tag{5}$$

where $R_1(\mathbf{x};\,t,\tau)$ is the shear relaxation function, $R_2(\mathbf{x};\,t,\tau)$ the volumetric relaxation function and δ_{ij} the Kronecker's symbol.

In the following, we shall refer only to constitutive law (1). It is supposed that the instantaneous relaxation kernel $R_{ijhk}(\mathbf{x};\,t,t)$ is positive definite, i.e.

$$R_{ijhk}(\mathbf{x};\,t,t)\gamma_{ij}\gamma_{hk} > 0 \tag{6}$$

for every $\mathbf{x}\in\Omega$ and $t_0 \leqslant t \leqslant +\infty$ and for every non-vanishing double symmetric tensor $\boldsymbol{\gamma}$.

2.2. The linear viscoelastic problem

Consider a linear viscoelastic body, occupying a region Ω of Euclidean tridimensional space with boundary surface Γ, where strains and displacements are small. The body is subjected to

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* The constitutive law is usually written in direct form as follows:

$$\sigma_{ij}(t) = \int_{-\infty}^{t} R_{ijhk}(t,\,\tau)\,\mathrm{d}\varepsilon_{hk}(\tau)$$

but, for the hypothesis of vanishing stresses and strains for any $t < t_0$, it becomes

$$\sigma_{ij}(t) = \int_{0^-}^{t} R_{ijhk}(t,\,\tau)\,\mathrm{d}\varepsilon_{hk}(\tau)$$

Pointing out the elastic part in $t = t_0$, we have (1). In the same way we obtain (2) a history of volume forces $F_i(\mathbf{x}; t)$, to a history of boundary forces $p_i(\mathbf{x}; t)$ acting on the loaded region Γ_p of the boundary, and to a history of imposed displacements $\bar{u}_i(\mathbf{x}; t)$ acting on the constrained region Γ_u (with $\Gamma = \Gamma_p \cup \Gamma_u$). The whole load history is supposed to be defined in the given time range T ($t \in T$ and $T = [t_0, t_1]$).

When mentioning 'viscoelastic problem' we refer to the problem of the determination of the viscoelastic response of the body (in terms of displacements $u_i(\mathbf{x}; t)$, strains $\varepsilon_{ij}(\mathbf{x}; t)$ and stresses $\sigma_{ij}(\mathbf{x}; t)$) under the above-mentioned actions, in every point $\mathbf{x}$ of the body and for every time t within the T interval.

The general linear viscoelastic problem is described by the following equations:

$$\sigma_{ij/j}(\mathbf{x}; t) + F_i(\mathbf{x}; t) = 0 \qquad \text{in } \Omega \times T \tag{7}$$

$$\sigma_{ij}(\mathbf{x}; t) n_j(\mathbf{x}) = p_i(\mathbf{x}; t) \qquad \text{on } \Gamma_p \times T \tag{8}$$

$$\varepsilon_{ij}(\mathbf{x}; t) = \tfrac{1}{2}[u_{i/j}(\mathbf{x}; t) + u_{j/i}(\mathbf{x}; t)] \qquad \text{in } \Omega \times T \tag{9}$$

$$u_i(\mathbf{x}; t) = \bar{u}_i(\mathbf{x}; t) \qquad \text{on } \Gamma_u \times T \tag{10}$$

$$\sigma_{ij}(\mathbf{x}; t) = R_{ijhk}(\mathbf{x}; t, t_0)\varepsilon_{hk}(\mathbf{x}; t_0) + \int_{t_0^+}^{t} R_{ijhk}(\mathbf{x}; t, \tau)\, \mathrm{d}\varepsilon_{hk}(\mathbf{x}; \tau) \qquad \text{in } \Omega \times T \tag{11}$$

where $(\cdot)/j \equiv \partial(\cdot)/\partial x_j$ and $n_j(\mathbf{x})$ is the outward unit vector normal to Γ at point $\mathbf{x}$.

Relations (7) and (8) represent the indefinite and boundary equilibrium equations, respectively. Relations (9) and (10) represent the indefinite and boundary compatibility equations, respectively, and relation (11) represents the constitutive law in direct form.

In the following, it is assumed that the problem (7)–(11) has unique solution. A unicity theorem can be found in Reference 13. In this paper, the existence of the viscoelastic solution will not be discussed.

3. AN EXTREMAL PRINCIPLE

Let us consider the following problem that we will name 'auxiliary':

$$\hat{\sigma}_{ij/j} + \hat{F}_i = 0 \qquad \text{in } \Omega \times T \tag{12}$$

$$\hat{\sigma}_{ij} n_j = \hat{p}_i \qquad \text{on } \Gamma_p \times T \tag{13}$$

$$\hat{\varepsilon}_{ij} = \tfrac{1}{2}(\hat{u}_{i/j} + \hat{u}_{j/i}) \qquad \text{in } \Omega \times T \tag{14}$$

$$\hat{u}_i = \bar{u}_i \qquad \text{on } \Gamma_u \times T \tag{15}$$

$$\hat{\sigma}_{ij} = R_{ijhk}(\mathbf{x}; t_0, t_0)\hat{\varepsilon}_{hk} \qquad \text{in } \Omega \times T \tag{16}$$

where the fields of displacement $\hat{u}_i$, of strain $\hat{\varepsilon}_{ij}$ and of stress $\hat{\sigma}_{ij}$, are the solution of the auxiliary problem and where $\hat{F}_i$ and $\hat{p}_i$ are volume and surface forces, respectively, that depend on the unknown actual displacement field u_i in the following way:

$$\hat{F}_i(u_i) = -\frac{\partial}{\partial x_j}\left[R_{ijhk}(\mathbf{x}; t, t_0)\frac{1}{2}(u_{h/k} + u_{k/h}) + \int_{t_0^+}^{t} R_{ijhk}(\mathbf{x}; t, \tau)\frac{1}{2}\mathrm{d}(u_{h/k} + u_{k/h})\right] \tag{17}$$

$$\hat{p}_i(u_i) = n_j\left[R_{ijhk}(\mathbf{x}; t, t_0)\frac{1}{2}(u_{h/k} + u_{k/h}) + \int_{t_0^+}^{t} R_{ijhk}(\mathbf{x}; t, \tau)\frac{1}{2}\mathrm{d}(u_{h/k} + u_{k/h})\right] \tag{18}$$

Let us now consider the functional:

$$\mathfrak{F}[u_i^*(\mathrm{x};t)] = \frac{1}{2}\int_{t_0}^{t_1}\int_\Omega R_{ijhk}(\mathrm{x};t_0,t_0)\hat{\varepsilon}_{ij}^*(\mathrm{x};t)\hat{\varepsilon}_{hk}^*(\mathrm{x};t)\,\mathrm{d}\Omega\,\mathrm{d}t - \int_{t_0}^{t_1}\int_\Omega F_i(\mathrm{x};t)\hat{u}_i^*(\mathrm{x};t)\,\mathrm{d}\Omega\,\mathrm{d}t - \int_{t_0}^{t_1}\int_{\Gamma_p} p_i(\mathrm{x};t)\hat{u}_i^*(\mathrm{x};t)\,\mathrm{d}\Gamma\,\mathrm{d}t \tag{19}$$

under the conditions

$$\varepsilon_{ij}^*(\mathrm{x};t) = \tfrac{1}{2}(u_{i/j}^* + u_{j/i}^*) \quad \text{in } \Omega\times T \tag{20}$$

$$u_i^*(\mathrm{x};t) = \bar{u}_i \quad \text{on } \Gamma_u\times T \tag{21}$$

where $\hat{\sigma}_{ij}^*$, $\hat{\varepsilon}_{ij}^*$ and $\hat{u}_i^*$ are the solution to the auxiliary problem (12)–(16) in which the loads $\hat{F}_i$ and $\hat{p}_i$ are substituted with the loads $\hat{F}_i^*$ and $\hat{p}_i^*$:

$$\hat{F}_i^*(u_i^*) = -\frac{\partial}{\partial x_j}\left[R_{ijhk}(\mathrm{x};t,t_0)\frac{1}{2}(u_{h/k}^* + u_{k/h}^*) + \int_{t_0^+}^{t} R_{ijhk}(\mathrm{x};t,\tau)\frac{1}{2}\mathrm{d}(u_{h/k}^* + u_{k/h}^*)\right] \tag{22}$$

$$\hat{p}_i^*(u_i^*) = n_j\left[R_{ijhk}(\mathrm{x};t,t_0)\frac{1}{2}(u_{h/k}^* + u_{k/h}^*) + \int_{t_0^+}^{t} R_{ijhk}(\mathrm{x};t,\tau)\frac{1}{2}\mathrm{d}(u_{h/k}^* + u_{k/h}^*)\right] \tag{23}$$

Proposition. Among the 'admissible' displacement fields $u_i^*(\mathrm{x};t)$, the solution of the viscoelastic problem, in the given time interval T, is the field that makes the functional (19) *minimum. The 'admissible' displacement fields are intended as those that respect the conditions* (20) *and* (21).

Proof. Let us consider the difference $\Delta\mathfrak{F}$ between the functional, valued for an arbitrary admissible displacement field, and the functional valued for the actual displacements field.

$$\Delta\mathfrak{F} = \mathfrak{F}[u_i^*(\mathrm{x};t)] - \mathfrak{F}[u_i(\mathrm{x};t)] \tag{24}$$

With the positions:

$$\Delta\hat{u}_i = \hat{u}_i^* - \hat{u}_i \tag{25}$$

$$\Delta\hat{\varepsilon}_{ij} = \hat{\varepsilon}_{ij}^* - \hat{\varepsilon}_{ij} \tag{26}$$

relation (24) becomes

$$\Delta\mathfrak{F} = \frac{1}{2}\int_{t_0}^{t_1}\int_\Omega \Delta\hat{\varepsilon}_{ij} R_{ijhk}(\mathrm{x};t_0,t_0)\Delta\hat{\varepsilon}_{hk}\,\mathrm{d}\Omega\,\mathrm{d}t + \int_{t_0}^{t_1}\int_\Omega \hat{\varepsilon}_{ij} R_{ijhk}(\mathrm{x};t_0,t_0)\Delta\hat{\varepsilon}_{hk}\,\mathrm{d}\Omega\,\mathrm{d}t - \int_{t_0}^{t_1}\int_\Omega F_i\Delta\hat{u}_i\,\mathrm{d}\Omega\,\mathrm{d}t - \int_{t_0}^{t_1}\int_{\Gamma_p} p_i\Delta\hat{u}_i\,\mathrm{d}\Gamma\,\mathrm{d}t \tag{27}$$

But, for the virtual work principle, the last three integrals vanish and therefore

$$\Delta\mathfrak{F} = \frac{1}{2}\int_{t_0}^{t_1}\int_\Omega \Delta\hat{\varepsilon}_{ij} R_{ijhk}(\mathrm{x};t_0,t_0)\Delta\hat{\varepsilon}_{hk}\,\mathrm{d}\Omega\,\mathrm{d}t \tag{28}$$

For the assumption (6), the elastic tensor $R_{ijhk}(\mathbf{x}; t_0, t_0)$ is positive definite and thus we have

$$\Delta\mathfrak{F} \geqslant 0 \tag{29}$$

$$\Delta\mathfrak{F} = 0 \quad \text{if and only if } \hat{\varepsilon}^*_{ij} \equiv \hat{\varepsilon}_{ij} \tag{30}$$

but relation (30) is valid if and only if

$$\hat{F}^*_i = \hat{F}_i \tag{31}$$

$$\hat{p}^*_i = \hat{p}_i \tag{32}$$

and this implies

$$\frac{\partial}{\partial x_j}\left[R_{ijhk}(\mathbf{x}; t, t_0)\frac{1}{2}(u^*_{h/k} + u^*_{k/h}) + \int_{t_0^+}^{t} R_{ijhk}(\mathbf{x}; t, \tau)\frac{1}{2}\mathrm{d}(u^*_{h/k} + u^*_{k/h})\right] + F_i = 0 \quad \text{in } \Omega\times T \tag{33}$$

$$\left[R_{ijhk}(\mathbf{x}; t, t_0)\frac{1}{2}(u^*_{h/k} + u^*_{k/h}) + \int_{t_0^+}^{t} R_{ijhk}(\mathbf{x}; t, \tau)\frac{1}{2}\mathrm{d}(u^*_{h/k} + u^*_{k/h})\right]n_j = p_i \quad \text{on } \Gamma_p\times T \tag{34}$$

which with the relations (20), (21) and the constitutive law give the whole equations set of the given problem. In the above proof it has been demonstrated that a solution of problem (7)–(11) is also a solution of the minimum problem. If the solution of the problem exists, this statement can also be inverted. In other words, the solution of the minimum problem is also solution of the problem (7)–(11). This is obvious for the supposed uniqueness of the solution and for the convexity of the functional (19).

3.1. Particularization of the functional to plane frame structures

The classical hypothesis of Bernoulli–Navier and first-order beam theories are assumed.

Let the kinematic of a straight beam be defined by the vector $\mathbf{u} = [u \;\; v]^{\mathrm{T}}$ where u and v are the x and y displacements of the centroid of the cross section at x (x is the longitudinal axis of the beam). Denoting with $\mathbb{N}(x; t)$ and $\mathbb{M}(x; t)$ the axial force and the bending moment, the constitutive viscoelastic law may be written

$$\mathbb{N}(x; t) = R(x; t, t_0)A(x)\varepsilon(x; t) + \int_{t_0^+}^{t} R(x; t, \tau)A(x)\,\mathrm{d}\varepsilon(x; \tau) \tag{35}$$

$$\mathbb{M}(x; t) = R(x; t, t_0)J(x)\chi(x; t) + \int_{t_0^+}^{t} R(x; t, \tau)J(x)\,\mathrm{d}\chi(x; \tau) \tag{36}$$

where $\varepsilon = \partial u/\partial x$ and $\chi = \partial^2 v/\partial x^2$ are the axial strain and curvature, while $A(x)$, $J(x)$ and $R(x; t, \tau)$ are the cross-section area, its moment of inertia and the relaxation function, respectively. Then, denoting with p, q, the longitudinal and transversal components of the distributed load, the general beam equilibrium equations become

$$-\frac{\partial}{\partial x}\left(R(x; t, t_0)A(x)\frac{\partial u(x; t)}{\partial x}\right) - \frac{\partial}{\partial x}\int_{t_0^+}^{t}\left(R(x; t, \tau)A(x)\frac{\partial \mathrm{d}u(x; \tau)}{\partial x}\right) = p \tag{37}$$

$$\frac{\partial^2}{\partial x^2}\left(R(x; t, t_0)J(x)\frac{\partial^2 v(x; t)}{\partial x^2}\right) + \frac{\partial^2}{\partial x^2}\int_{t_0^+}^{t}\left(R(x; t, \tau)J(x)\frac{\partial^2 \mathrm{d}v(x; \tau)}{\partial x^2}\right) = q \tag{38}$$

When the structure is subdivided in a number n^e of straight elements (the index e is added to any quantity relevant to the generic element e) the extended total potential energy functional for the whole of the structure becomes (in the following, the symbol $*$ indicating all 'admissible quantities' will be omitted for simplicity):

$$\mathfrak{F}[u, v] = \sum_{e=1}^{n^e} \left\{ \frac{1}{2} \int_{t_0}^{t_1} \int_0^{l^e} R^e(x; t_0, t_0) \left[A^e(x) \left(\frac{\partial \hat{u}^e(x; t)}{\partial x} \right)^2 + J^e(x) \left(\frac{\partial^2 \hat{v}^e(x; t)}{\partial x^2} \right)^2 \right] \mathrm{d}x\, \mathrm{d}t - \int_{t_0}^{t_1} \int_0^{l^e} p^e(x; t) \hat{u}^e(x; t)\, \mathrm{d}x\, \mathrm{d}t - \int_{t_0}^{t_1} \int_0^{l^e} q^e(x; t) \hat{v}^e(x; t)\, \mathrm{d}x\, \mathrm{d}t \right\} \tag{39}$$

where $\hat{u}^e$, $\hat{v}^e$ are the 'fictitious' elastic displacements for the element e. These 'fictitious' displacements are the elastic response of the structure (with flexural stiffness $R^e(x; t_0, t_0) J^e(x)$ and with axial stiffness $R^e(x; t_0, t_0) A^e(x)$) due to the imposed displacements $\bar{u}^{ej}$, $\bar{v}^{ej}$, $\bar{\vartheta}^{ej}$ ($\bar{\vartheta}^{ej}$ is the imposed rotation at node j of element e) and to the longitudinal and transversal 'fictitious' loads $\hat{p}^e$ and $\hat{q}^e$:

$$\hat{p}^e = -\frac{\partial}{\partial x} \left(R^e(x; t, t_0) A^e(x) \frac{\partial u^e(x; t)}{\partial x} \right) - \frac{\partial}{\partial x} \int_{t_0^+}^{t} R^e(x; t, \tau) A^e(x) \frac{\partial \mathrm{d}u^e(x; \tau)}{\partial x} \tag{40}$$

$$\hat{q}^e = \frac{\partial^2}{\partial x^2} \left(R^e(x; t, t_0) J^e(x) \frac{\partial^2 v^e(x; t)}{\partial x^2} \right) + \frac{\partial^2}{\partial x^2} \int_{t_0^+}^{t} R^e(x; t, \tau) J^e(x) \frac{\partial^2 \mathrm{d}v^e(x; \tau)}{\partial x^2} \tag{41}$$

4. SPACE AND TIME DISCRETIZED FORMULATION FOR PLANE FRAME STRUCTURES

4.1. Spatial discretization

In the following, for simplicity, we will assume having vanishing displacements on Γ_u. Finite elements with two nodes will be considered, with three degrees of freedom in each node (Figure 1), with linear shape functions for the 'extensional' degrees of freedom r_1 and r_2 and with cubic shape functions for the 'bending' degrees of freedom r_3, r_4, r_5, r_6.

The shape functions n_i ($i = 1, \ldots, 6$) relevant to the degrees of freedom $1, \ldots, 6$ are, respectively,

$$\begin{aligned} n_1(x) &= 1 - x/l \\ n_2(x) &= x/l \\ n_3(x) &= 1 - 3x^2/l^2 + 2x^3/l^3 \\ n_4(x) &= l(x/l - 2x^2/l^2 - x^3/l^3) \\ n_5(x) &= 3x^2/l^2 - 2x^3/l^3 \\ n_6(x) &= l(x^3/l^3 - x^2/l^2) \end{aligned} \tag{42}$$

In the following we will suppose for simplicity that each element has constant cross-section (i.e. A^e and J^e independent of x) and homogeneous material (i.e. $R^e = R^e(t, \tau)$).

Let $\mathbf{u}^e(x; t)$ be the displacement vector, in local co-ordinates, of an internal point x of the finite element e, whose u^e is the displacement component along axis x and v^e is the component along axis y:

$$\mathbf{u}^e(x; t) = \begin{bmatrix} u^e \\ v^e \end{bmatrix} \tag{43}$$

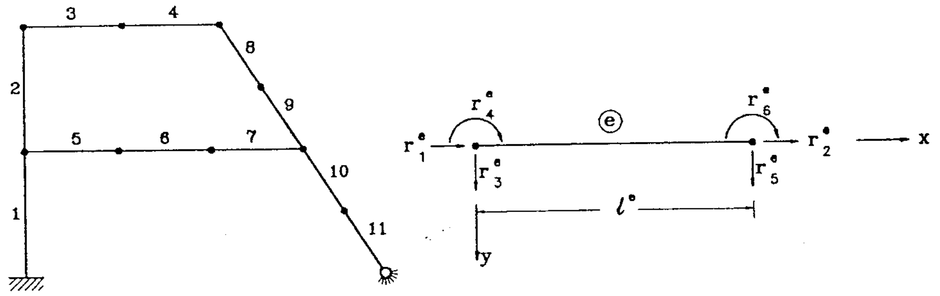


Figure 1. Finite elements spatial discretization of a plane frame

Collecting the shape functions $n_1(x)$, $n_2(x)$ into the vector $\mathbf{n}_u^e$ and the functions $n_3(x), \ldots, n_6(x)$ into the vector $\mathbf{n}_v^e$, one can express the displacement vector as a function of the nodal degrees of freedom $r_1, \ldots, r_6$ as follows:

$$\mathbf{u}^e(x;t) = \begin{bmatrix} \mathbf{n}_u^{eT} & \mathbf{0}^T \\ \mathbf{0}^T & \mathbf{n}_v^{eT} \end{bmatrix} \begin{bmatrix} \mathbf{r}_u^e \\ \mathbf{r}_v^e \end{bmatrix} = \mathbf{N}^e(x)\,\mathbf{r}^e(t) \tag{44}$$

where

$$\begin{aligned} \mathbf{r}_u^e &= [r_1^e \ r_2^e]^T \\ \mathbf{r}_v^e &= [r_3^e \ r_4^e \ r_5^e \ r_6^e]^T \end{aligned} \tag{45}$$

Denoting with $\boldsymbol{\alpha}(t) = [\alpha_1 \ \alpha_2 \ \ldots \ \alpha_{ns}]^T$ the vector of the degrees of freedom of the assembled structure, with respect to the global reference system, one obtains

$$\mathbf{r}_u^e(t) = \mathbf{A}_u^e \boldsymbol{\alpha}(t); \quad \mathbf{r}_v^e(t) = \mathbf{A}_v^e \boldsymbol{\alpha}(t) \tag{46}$$

$$\mathbf{r}^e(t) = \mathbf{A}^e \boldsymbol{\alpha}(t) \tag{47}$$

where

$$\mathbf{A}^e = \begin{bmatrix} \mathbf{A}_u^e \\ \mathbf{A}_v^e \end{bmatrix} = \mathbf{C}^e \mathbf{T}^e \tag{48}$$

is the product between the Boole's connectivity matrix $\mathbf{C}^e$ and the co-ordinate transformation matrix $\mathbf{T}^e$.

4.2. Time discretization

The vector $\boldsymbol{\alpha}(t)$ of the degrees of freedom of the assembled structure is now written as a function of time degrees of freedom $\boldsymbol{\beta} = [\beta_1 \ \beta_2 \ \ldots \ \beta_N]^T$ through shape functions collected into the matrix $\mathbf{M}(t)$, with $t \in T$

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{ns} \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1^T & & \\ & \ddots & \\ & & \mathbf{m}_{ns}^T \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{\beta}_1 \\ \vdots \\ \boldsymbol{\beta}_{ns} \end{bmatrix} = \mathbf{M}(t)\boldsymbol{\beta} \tag{49}$$

In every example of Section 5 all the spatial degrees of freedom $\alpha_i(t)$ $(1 \leqslant i \leqslant ns)$ are discretized with respect to time using the same shape functions, i.e.

$$\mathbf{m}_1 = \cdots = \mathbf{m}_{ns} = \mathbf{m} = [m_1 \ \ldots \ m_{nt}]^T \tag{50}$$

In this way the total number of degrees of freedom is $N = ns \cdot nt$.

4.3. *Determination of* $\hat{\mathbf{u}}$

The vector $\hat{\mathbf{u}}$ may be obtained through the stationarity of the total potential energy

$$F_{\text{tpe}}[\hat{u}, \hat{v}] = \sum_{e=1}^{n^e} \left\{ \frac{1}{2} \int_0^{l^e} R^e(t_0, t_0) A^e \left(\frac{\partial \hat{u}^e}{\partial x} \right)^2 \mathrm{d}x + \frac{1}{2} \int_0^{l^e} R^e(t_0, t_0) J^e \left(\frac{\partial^2 \hat{v}^e}{\partial x^2} \right)^2 \mathrm{d}x - \int_0^{l^e} \hat{u}^e \hat{p}^e \, \mathrm{d}x - \int_0^{l^e} \hat{v}^e \hat{q}^e \, \mathrm{d}x \right\} \tag{51}$$

It is possible to discretize $\hat{\mathbf{u}}^e$ in the same way as $\mathbf{u}^e$, i.e.

$$\hat{\mathbf{u}}^e = \mathbf{N}^e(x) \mathbf{A}^e \hat{\boldsymbol{\alpha}}(t) \tag{52}$$

with $\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$.

By substituting equation (52) into the functional (51), we have

$$F_{\text{tpe}}[\hat{u}, \hat{v}] = \frac{1}{2} \hat{\boldsymbol{\alpha}}^{\mathrm{T}} \left\{ \sum_{e=1}^{n^e} R^e(t_0, t_0) A^e \mathbf{A}_u^{e\mathrm{T}} \left(\int_0^{l^e} \frac{\mathrm{d}\mathbf{n}_u^e}{\mathrm{d}x} \left(\frac{\mathrm{d}\mathbf{n}_u^e}{\mathrm{d}x} \right)^{\mathrm{T}} \mathrm{d}x \right) \mathbf{A}_u^e + \sum_{e=1}^{n^e} R^e(t_0, t_0) J^e \mathbf{A}_v^{e\mathrm{T}} \left(\int_0^{l^e} \frac{\mathrm{d}^2\mathbf{n}_v^e}{\mathrm{d}x^2} \left(\frac{\mathrm{d}^2\mathbf{n}_v^e}{\mathrm{d}x^2} \right)^{\mathrm{T}} \mathrm{d}x \right) \mathbf{A}_v^e \right\} \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\alpha}}^{\mathrm{T}} \left\{ \sum_{e=1}^{n^e} \mathbf{A}_u^{e\mathrm{T}} \int_0^{l^e} \mathbf{n}_u^e \hat{p}^e \, \mathrm{d}x + \sum_{e=1}^{n^e} \mathbf{A}_v^{e\mathrm{T}} \int_0^{l^e} \mathbf{n}_v^e \hat{q}^e \, \mathrm{d}x \right\} \tag{53}$$

The stationarity of (53) leads to the following systems of equations:

$$\mathbf{K} \hat{\boldsymbol{\alpha}} = \hat{\mathbf{P}} \tag{54}$$

where

$$\mathbf{K} = \sum_{e=1}^{n^e} \mathbf{A}^{e\mathrm{T}} \mathbf{k}^e \mathbf{A}^e \tag{55}$$

$$\mathbf{k}^e = \int_0^{l^e} \begin{bmatrix} R^e(t_0, t_0) A^e \dfrac{\mathrm{d}\mathbf{n}_u^e}{\mathrm{d}x} \left(\dfrac{\mathrm{d}\mathbf{n}_u^e}{\mathrm{d}x} \right)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & R^e(t_0, t_0) J^e \dfrac{\mathrm{d}^2\mathbf{n}_v^e}{\mathrm{d}x^2} \left(\dfrac{\mathrm{d}^2\mathbf{n}_v^e}{\mathrm{d}x^2} \right)^{\mathrm{T}} \end{bmatrix} \mathrm{d}x \tag{56}$$

$$\hat{\mathbf{P}} = \sum_{e=1}^{n^e} \mathbf{A}^{e\mathrm{T}} \int_0^{l^e} \begin{bmatrix} \mathbf{n}_u^e \hat{p}^e \\ \mathbf{n}_v^e \hat{q}^e \end{bmatrix} \mathrm{d}x \tag{57}$$

and $\mathbf{K}$ is the stiffness matrix of the assembled structure. By substitution of $\hat{p}^e$ and $\hat{q}^e$ (equations (40) and (41)) into the second member of equation (57) we obtain

$$\hat{\mathbf{P}} = \sum_{e=1}^{n^e} \mathbf{A}^{e\mathrm{T}} \int_0^{l^e} \begin{bmatrix} -\mathbf{n}_u^e \left(R^e(t, t_0) A^e \dfrac{\partial^2 u^e(x; t)}{\partial x^2} + \displaystyle\int_{t_0^+}^{t} R^e(t, \tau) A^e \dfrac{\partial^2 \mathrm{d}u^e(x; \tau)}{\partial x^2} \right) \\ \mathbf{n}_v^e \left(R^e(t, t_0) J^e \dfrac{\partial^4 v^e(x; t)}{\partial x^4} + \displaystyle\int_{t_0^+}^{t} R^e(t, \tau) J^e \dfrac{\partial^4 \mathrm{d}v^e(x; \tau)}{\partial x^4} \right) \end{bmatrix} \mathrm{d}x \tag{58}$$

On integration by parts of equation (58), taking into account interelement compatibility and the imposed vanishing boundary constraint, we have

$$\hat{\mathbf{P}} = \sum_{e=1}^{n^e} \mathbf{A}^{e\mathrm{T}} \int_0^{l^e} \begin{bmatrix} \dfrac{\mathrm{d}\mathbf{n}_u^e}{\mathrm{d}x}\left(R^e(t, t_0)\, A^e \dfrac{\partial u^e(x;t)}{\partial x} + \displaystyle\int_{t_0^+}^{t} R^e(t,\tau)\, A^e \dfrac{\partial \mathrm{d}u^e(x;\tau)}{\partial x}\right) \\ \dfrac{\mathrm{d}^2\mathbf{n}_v^e}{\mathrm{d}x^2}\left(R^e(t, t_0)\, J^e \dfrac{\partial^2 v^e(x;t)}{\partial x^2} + \displaystyle\int_{t_0^+}^{t} R^e(t,\tau)\, J^e \dfrac{\partial^2 \mathrm{d}v^e(x;\tau)}{\partial x^2}\right) \end{bmatrix} \mathrm{d}x \tag{59}$$

which transforms, taking into account equations (44), (47) and (49), into

$$\hat{\mathbf{P}} = \sum_{e=1}^{n^e} \mathbf{A}^{e\mathrm{T}} \left(\int_0^{l^e} \begin{bmatrix} R^e(t_0, t_0)\, A^e \dfrac{\mathrm{d}\mathbf{n}_u^e}{\mathrm{d}x}\left(\dfrac{\mathrm{d}\mathbf{n}_u^e}{\mathrm{d}x}\right)^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & R^e(t_0, t_0)\, J^e \dfrac{\mathrm{d}^2\mathbf{n}_v^e}{\mathrm{d}x^2}\left(\dfrac{\mathrm{d}^2\mathbf{n}_v^e}{\mathrm{d}x^2}\right)^{\mathrm{T}} \end{bmatrix} \mathrm{d}x\, \mathbf{A}^e \right)$$

$$\times \left[\frac{R^e(t, t_0)}{R^e(t_0, t_0)} \mathbf{M}(t_0) + \int_{t_0^+}^{t} \frac{R^e(t,\tau)}{R^e(t_0, t_0)} \mathrm{d}\mathbf{M}(\tau) \right] \boldsymbol{\beta} =$$

$$= \sum_{e=1}^{n^e} \mathbf{A}^{e\mathrm{T}} \mathbf{k}^e \mathbf{A}^e \mathbf{R}^e \boldsymbol{\beta} = \mathbf{H}\boldsymbol{\beta} \tag{60}$$

where

$$\mathbf{R}^e = \frac{R^e(t, t_0)}{R^e(t_0, t_0)} \mathbf{M}(t_0) + \int_{t_0^+}^{t} \frac{R^e(t,\tau)}{R^e(t_0, t_0)} \mathrm{d}\mathbf{M}(\tau) \tag{61}$$

4.4. Discretized functional

Now it is possible to rewrite the extended energy functional (39) in a discretized form by using the spatial and time approximation introduced in Sections 4.1 and 4.2 and the calculation of Section 4.3:

$$\mathfrak{F}[\boldsymbol{\beta}] = \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}} \left\{ \int_{t_0}^{t_1} \mathbf{H}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{H}\, \mathrm{d}t \right\} \boldsymbol{\beta} - \boldsymbol{\beta}^{\mathrm{T}} \left\{ \int_{t_0}^{t_1} \mathbf{H}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{f}\, \mathrm{d}t \right\} = \frac{1}{2}\boldsymbol{\beta}^{\mathrm{T}}\mathbf{L}\boldsymbol{\beta} - \boldsymbol{\beta}^{\mathrm{T}}\mathbf{g} \tag{62}$$

where

$$\mathbf{f} = \sum_{e=1}^{n^e} \mathbf{A}^{e\mathrm{T}} \int_0^{l^e} \mathbf{N}^{e\mathrm{T}}\mathbf{p}^e\, \mathrm{d}x; \quad \mathbf{L} = \int_{t_0}^{t_1} \mathbf{H}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{H}\, \mathrm{d}t; \quad \mathbf{g} = \int_{t_0}^{t_1} \mathbf{H}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{f}\, \mathrm{d}t \tag{63}$$

and $\mathbf{p}^e = [p^e \;\; q^e]^{\mathrm{T}}$.

The minimum of $\mathfrak{F}[\boldsymbol{\beta}]$ is reached when $\boldsymbol{\beta}$ is the solution of the linear system:

$$\mathbf{L}\boldsymbol{\beta} = \mathbf{g} \tag{64}$$

L is the 'extended stiffness matrix' and **g** is the 'extended vector of the equivalent nodal forces'.

A flow-chart for the computer calculation of **L** and **g** is given in Figure 2.

Remark. Let us consider a non-singular linear equations system:

$$\mathbf{B}\mathbf{x} = \mathbf{b} \tag{65}$$

where **B** denotes a non-symmetric and non-definite real matrix $N \times N$ and $\mathbf{b} \in \mathbb{R}^N$. We can substitute the problem (65) with the 'normal equation' as in the least square method:

$$\mathbf{B}^{\mathrm{T}}\mathbf{B}\mathbf{x} = \mathbf{B}^{\mathrm{T}}\mathbf{b} \tag{66}$$

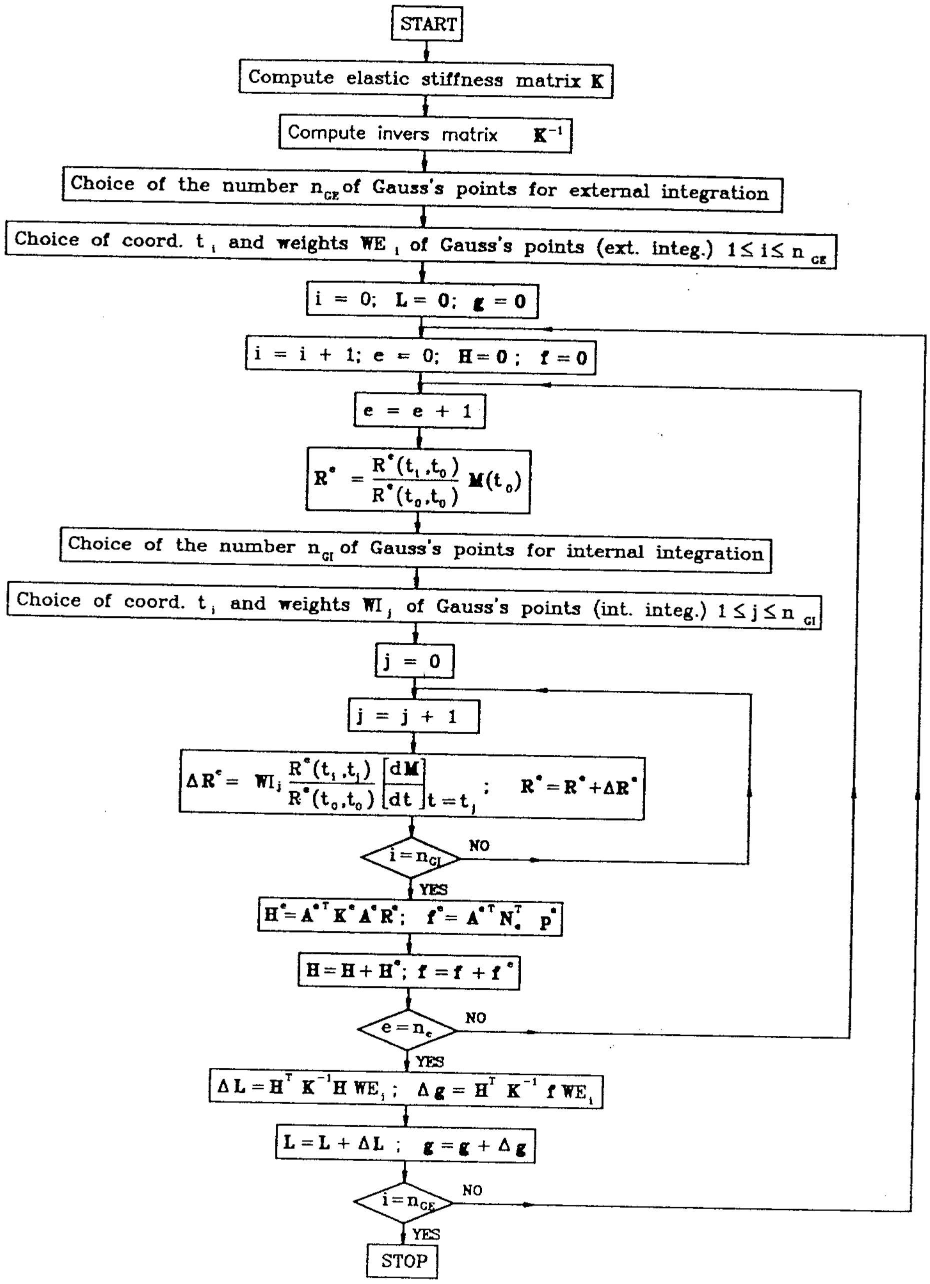


Figure 2. Flow chart for the calculation of matrix L and vector g

Thus the original problem (65) is replaced by the second one whose matrix is symmetric and semi-definite. Problem (66) is equivalent to the convex quadratic functional minimization.

Observing problem (66) it results clear that the main drawback of the least square method is the possible deterioration of the conditioning index which can make the solution sensitive to roundoff errors. Often this drawback can be overcome by substituting problem (65) not with the second one, but with the following:

$$\mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{B}\mathbf{x} = \mathbf{B}^{\mathrm{T}}\mathbf{S}\mathbf{b} \tag{67}$$

where S is a $N \times N$ symmetric and positive-definite matrix.[14]

With a suitable choice of S we can improve the conditioning of the matrix $\mathbf{B}^{\mathrm{T}}\mathbf{B}$. The matrix S can be considered as a 'scaling' (or 'preconditioning') matrix. It is evident that the structure of system (64) is of the same kind as system (67), where the preconditioning matrix is represented by $\mathbf{K}^{-1}$ which, as shown in Example d of the following section, results to be a good choice.

5. EXAMPLES

Here we introduce four examples of calculation of the structural effects of creep. For every example we use both the hereditary Kelvin–Voigt model[15–16] and the mixed CEB '78 model,[17]

The linear viscoelastic constitutive law (1)–(2) may be specialized for the mono-axial case as follows:

$$\sigma(t, t_0) = \varepsilon(t_0) R(t, t_0) + \int_{t_0^+}^{t} R(t, \tau)\, \mathrm{d}\varepsilon(\tau) \tag{68}$$

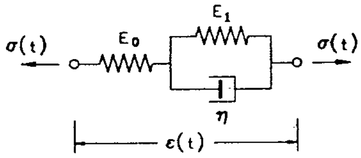
$$\varepsilon(t, t_0) = \sigma(t_0) \Phi(t, t_0) + \int_{t_0^+}^{t} \Phi(t, \tau)\, \mathrm{d}\sigma(\tau) \tag{69}$$

where

$R(t, \tau) =$ relaxation function

$\Phi(t, \tau) =$ creep function.

For the Kelvin–Voigt model the creep function Φ and the relaxation function R are



$$R(t, \tau) = R(t - \tau) = E_0 - \frac{E_0^2}{E_0 + E_1}\left(1 - \mathrm{e}^{-(E_0 + E_1)(t - \tau)/\eta}\right)$$
$$= E_\infty + (E_0 - E_\infty)\mathrm{e}^{-(t-\tau)/T^*} \tag{70}$$

$$\Phi(t, \tau) = \Phi(t - \tau) = \frac{1}{E_0} + \frac{1}{E_1}\left(1 - \mathrm{e}^{-E_1(t-\tau)/\eta}\right)$$
$$= \frac{1}{E_\infty} + \left(\frac{1}{E_0} + \frac{1}{E_\infty}\right)\mathrm{e}^{-(t-\tau)/\tau^*} \tag{71}$$

where

$$E_\infty = \frac{E_0 E_1}{E_0 + E_1}; \quad \tau^* = \frac{\eta}{E_1}; \quad T^* = \frac{\eta}{E_0 + E_1} \tag{72}$$

For the CEB'78 model we have

$$\Phi(t,\tau)=\frac{1}{E_c(\tau)}+\frac{\phi_{28}(t,\tau)}{E_{c28}}=\frac{1}{E_c(\tau)}+\frac{1}{E_{c28}}[\beta_a(\tau)+\phi_d\beta_d(t-\tau)+\phi_f[\beta_f(t)-\beta_f(\tau)]] \quad (73)$$

where

$1/E_c(\tau)$	is the initial elastic deformation (for a unit stress)
$\beta_a(\tau)/E_{c28}$	is the partially irreversible rapid initial deformation of the first day
$\phi_d\beta_d(t-\tau)/E_{c28}$	is the recoverable part of the delayed deformation ageing independent
$\phi_f[\beta_f(t)-\beta_f(\tau)]/E_{c28}$	is the irreversible delayed deformation ageing dependent ($\phi_f=\phi_{f1}\phi_{f2}$ in Appendix D of Reference 17)

The analytical expressions of various parameters may be found in Appendix D of Reference 17, as functions of the relative humidity RH and of the effective thickness h_0. The corresponding analytical formulation of the relaxation function is instead unknown to the authors.

In all the examples exponential time shape functions are used for the Kelvin–Voigt model:

$$\mathbf{m}^T=[1, e^{-t/t^*}, e^{-2t/t^*}, \ldots] \quad (74)$$

These shape functions proved as being the most suitable for the creep kernel with respect to polynomial or damped polynomial functions. The value of t^* is assumed in the range T^*–τ^*.

For the CEB'78 model logarithmic time shape functions are used:

$$\mathbf{m}^T=[1, \ln t, (\ln t)^2, \ldots] \quad (75)$$

that fit well the phenomenon on the entire field, with the exception of the rapidly increasing first day deformation.

For each example, significant displacements and/or stresses are plotted versus time, for the three intervals 28–100, 28–1000, 28–10000 d, and for 1, 2, 3 time degrees of freedom.

The time shape functions are defined within the entire interval T. In each diagram both the exact solution and the percentage error are also plotted. The exact solution is known in analytical form for the Kelvin–Voigt kernel and is obtained numerically for the CEB'78 kernel through the so-called step-by-step general method[17] (Examples a, b and c) or through program Abaqus[18] (Example d). For all diagrams the time axis is plotted using a logarithmic scale.

Examples a and b deal with the problem of homogeneous structures on elastic supports; Example c deals with homogeneous structures subjected to a modification of restraints and Example d deals with non-homogeneous structures. In all examples, integrations are made with the Gauss method using 20 Gauss points.

Remark. The asymptotic character of the viscoelastic response over large time interval justifies the above choice of the damped exponential (74) or logarithmic (75) shape functions which appear more suitable for good approximation of the structural behaviour even in the presence of a small number of degrees of freedom.

Polynomial shape functions can also be usefully used, devoting however particular attention to the wideness of the time interval which has to be suitably reduced in order to obtain, under the same number of degrees of freedom, the same kind of accuracy. This is easily confirmed by the plots of Figure 3 of Example a.

A wider discussion on the choice of the shape functions and of the number of degrees of freedom and a comparison of the relevant levels of accuracy obtained will play a fundamental role in a paper (Part II) in progress,[19] dealing with more complex 2-D and 3-D numerical applications.

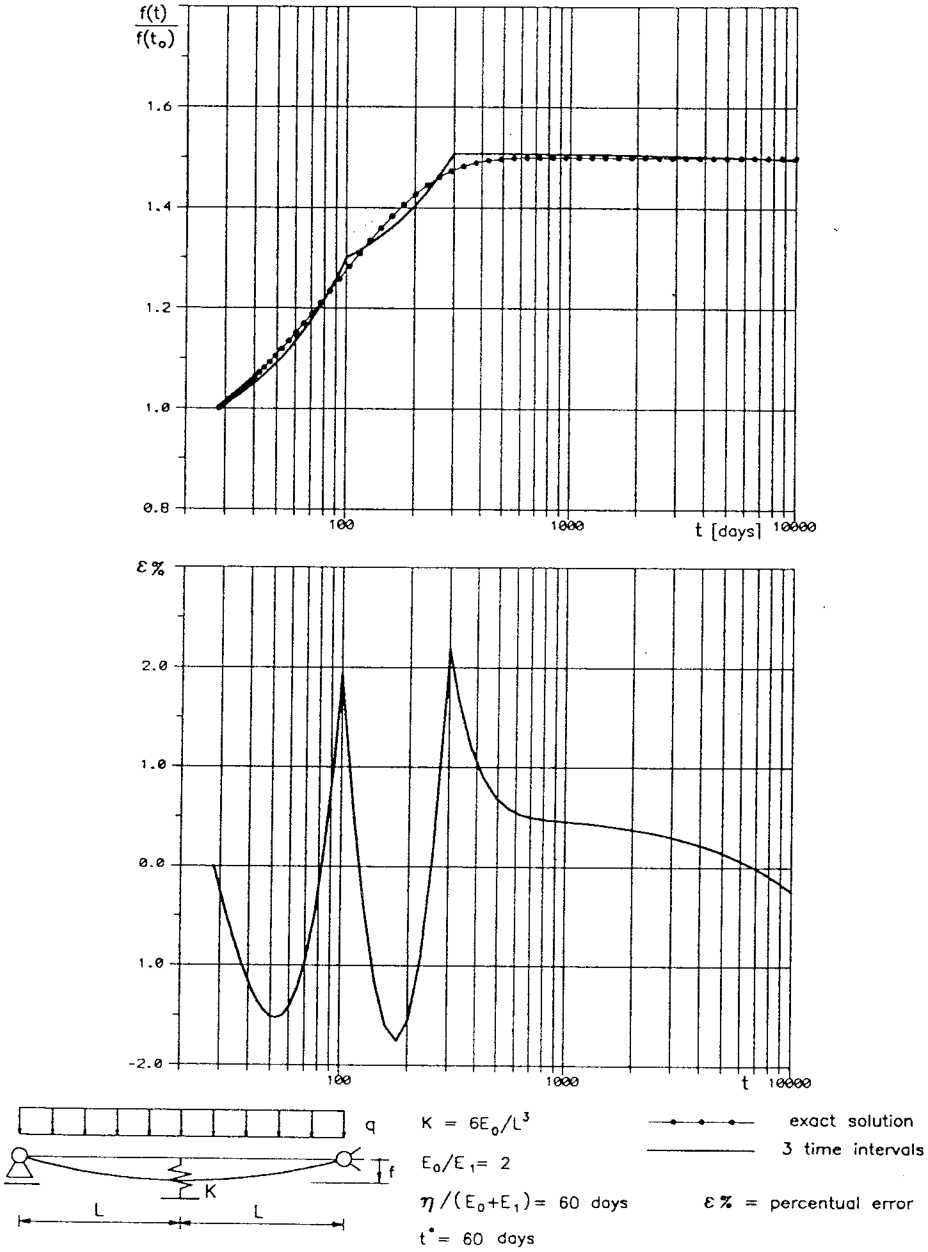


Figure 3. Example a: Beam on elastic support—Kelvin–Voigt. Use of polynomial shape functions applied on three time subintervals

Example a. The structure is a beam on rigid end supports, with an elastic support in the middle; the beam is homogeneous and has a constant cross-section. The uniformly distributed load is applied at time $t_0 = 28$ days and remains constant.

In Figures 4 and 5, the ratio between the displacement $f(t)$ of the middle section at time t and the initial elastic displacement $f(t_0)$ at time t_0 is plotted, both for the Kelvin–Voigt and the CEB'78 viscous kernel.

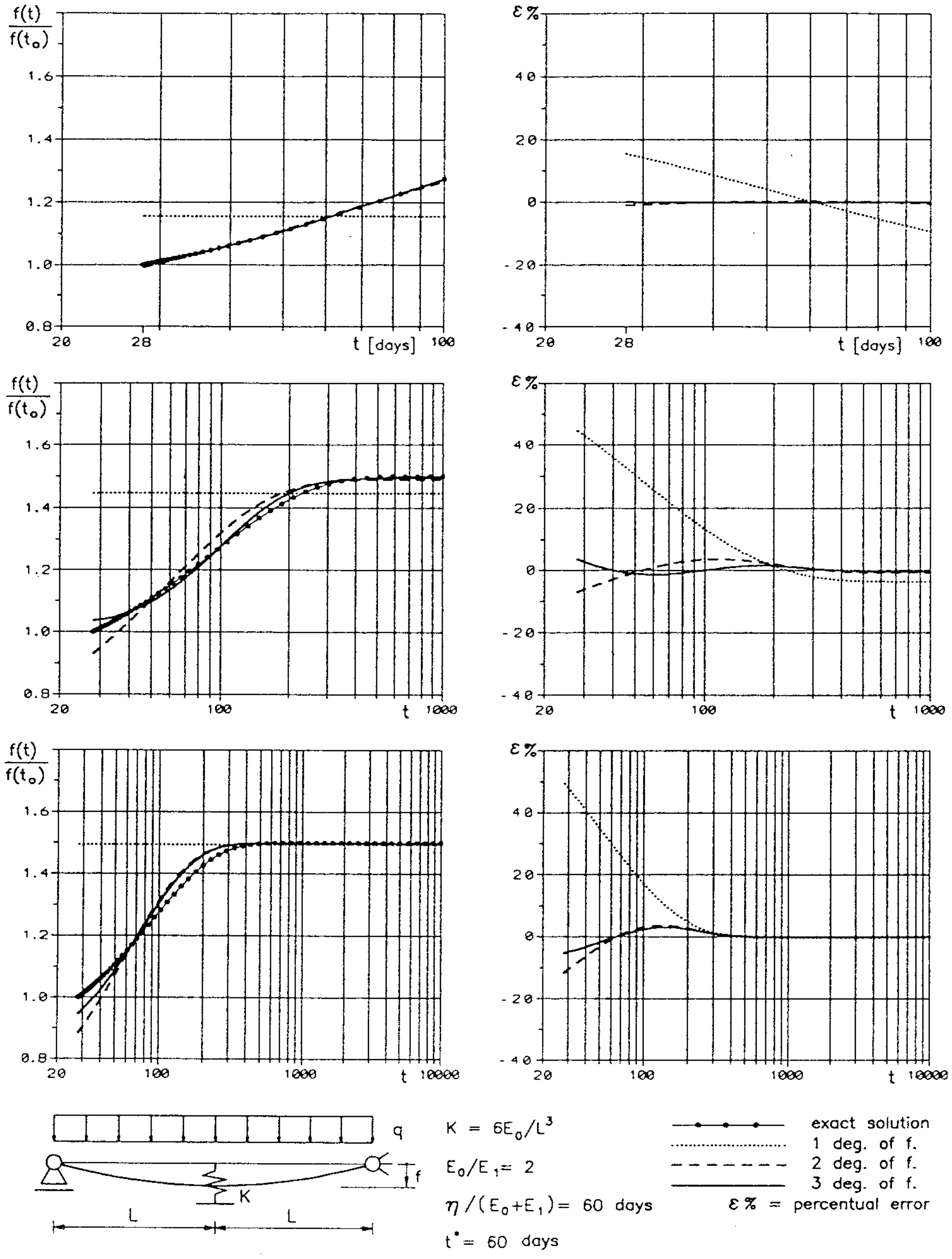


Figure 4. Example a: Beam on elastic support—Kelvin–Voigt

The diagrams show that the presented method allows a good interpretation of the viscoelastic behaviour of the structure. In the range t_0–10 000 the asymptotic final displacement is obtained with good accuracy even using one degree of freedom in time. With three degrees of freedom the solution is satisfactory within the entire range (the error is less than 3 per cent).

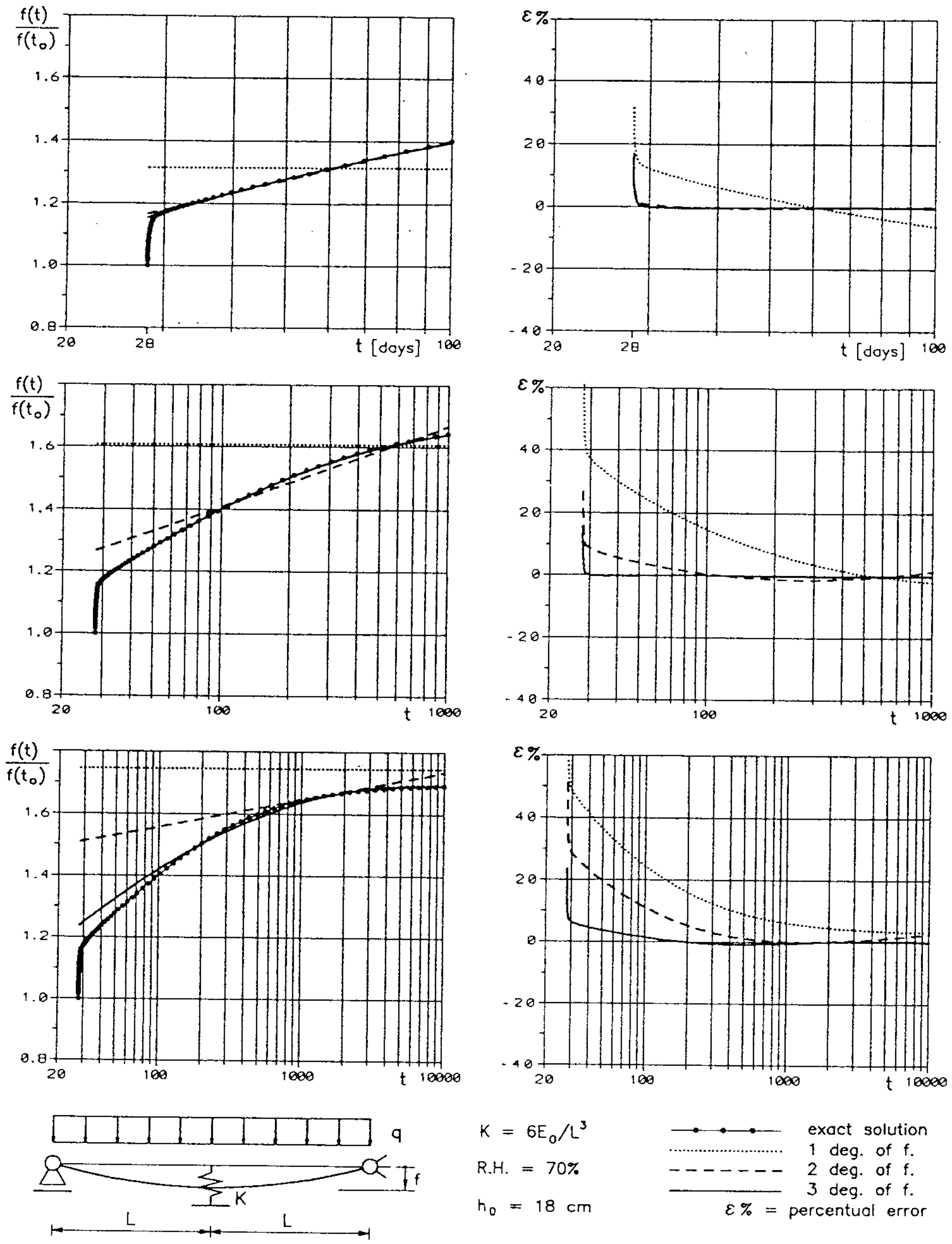


Figure 5. Example a: Beam on elastic support—CEB'78

In Figure 3 the ratio $f(t)/f(t_0)$ is plotted in the range t_0–10 000 for the Kelvin–Voigt kernel only. Here the integration interval was subdivided into three subintervals: 28–100, 100–300, 300–10 000 d. For each subinterval the proposed method was applied by using just a degree of freedom, i.e. by employing a linear time shape function with known initial value given by the end value of the previous interval. The percentage error is almost everywhere less than 2 per cent.

Example b. The same structure of Example a is here subjected to the constant displacement Δ of the spring end, applied at time $t_0 = 28$ d.

In Figures 6 and 7 the ratio between the bending moment in the central section $M(t)$ at time t and the corresponding initial elastic value $M(t_0)$ is plotted, both for the Kelvin–Voigt and the CEB'78 viscous kernel.

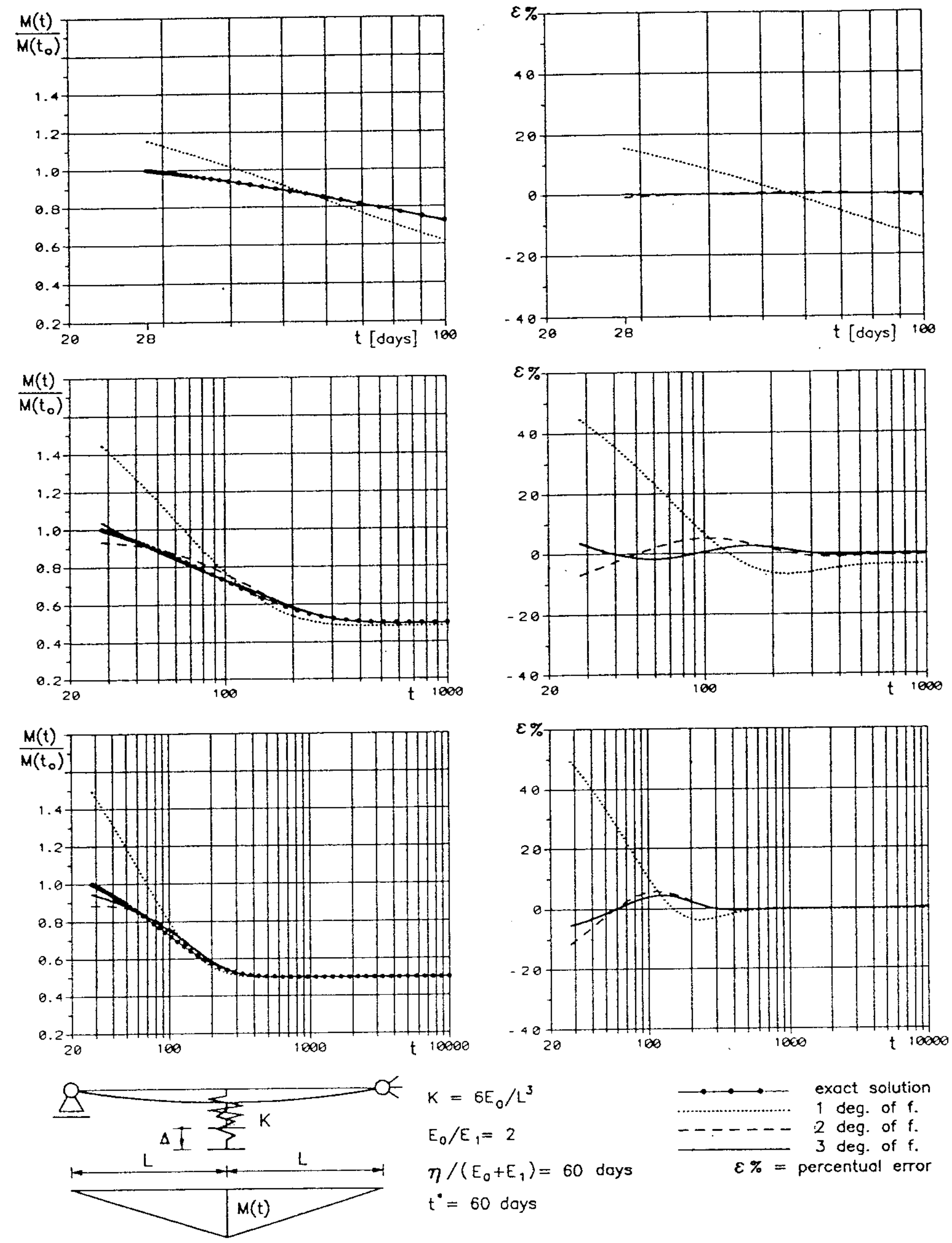


Figure 6. Example b: Beam on elastic support—Kelvin–Voigt

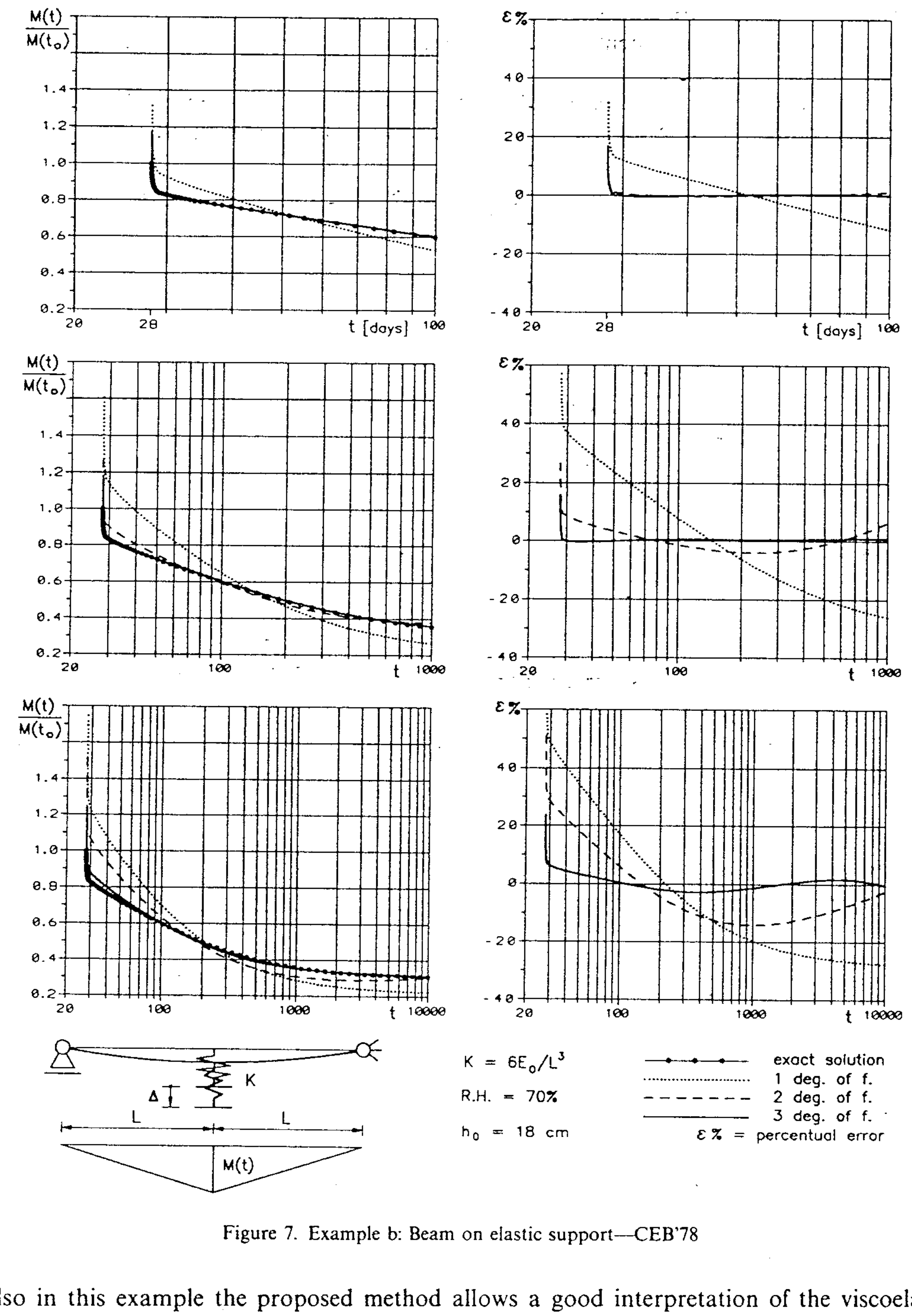


Figure 7. Example b: Beam on elastic support—CEB'78

Also in this example the proposed method allows a good interpretation of the viscoelastic behaviour of the structure. Errors are slightly greater than those of Example a, since the bending moment is proportional to the second derivative of the displacement and therefore is more sensitive to the approximation of the calculation. With three degrees of freedom in time the solution is satisfactory on the entire range t_0–10 000 anyway.

Example c. The structure consists of a two span continuous beam that is built introducing solidarity between two independent spans immediately after they have been subjected to load q at time t_0.

In Figures 8 and 9 the ratio between the bending moment at the mid support $M(t)$ at time t and the corresponding value $ql^2/8$ for the classic elastic continuous beam is plotted. The percentage

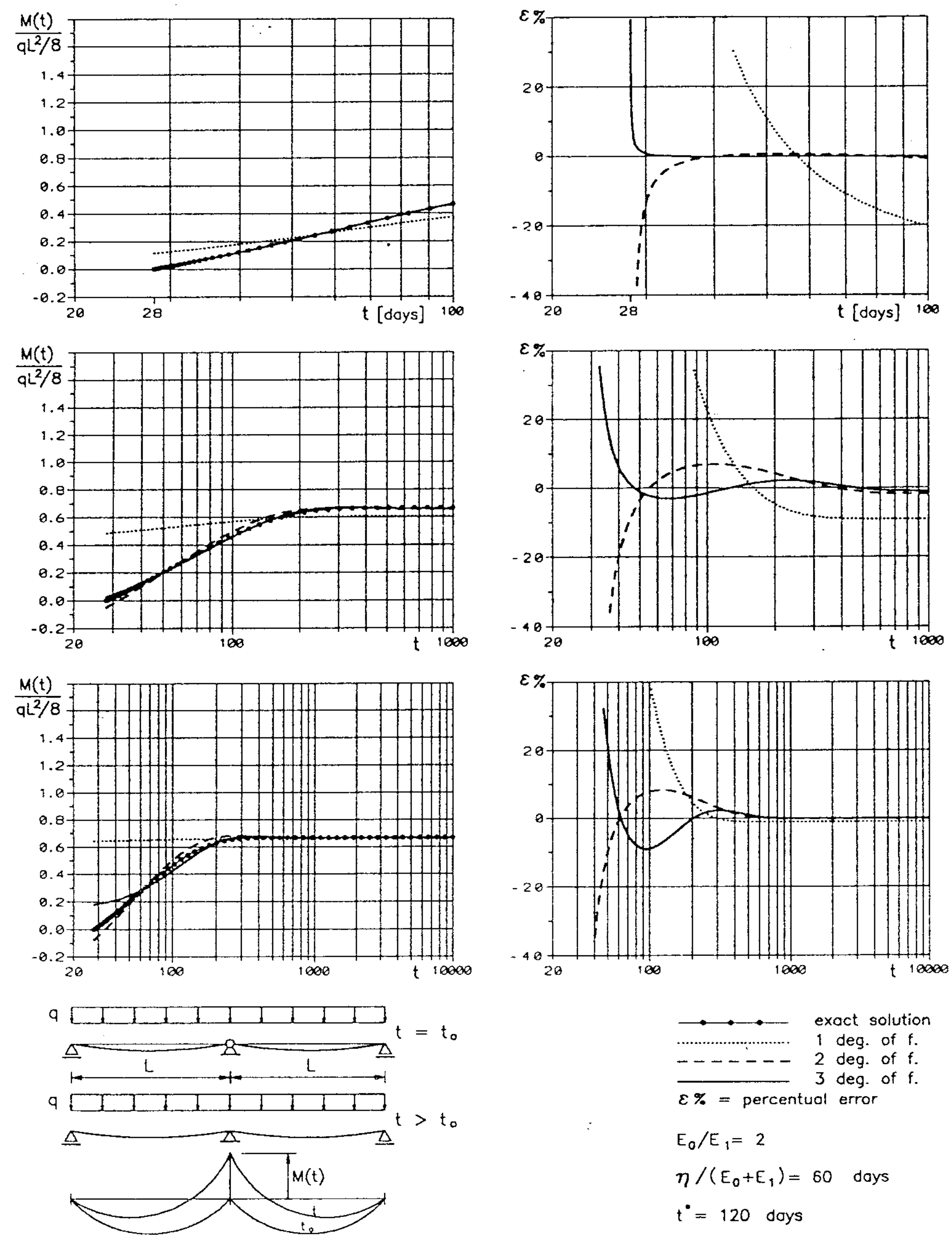


Figure 8. Example c: Beam subjected to a modification of restraint—Kelvin–Voigt

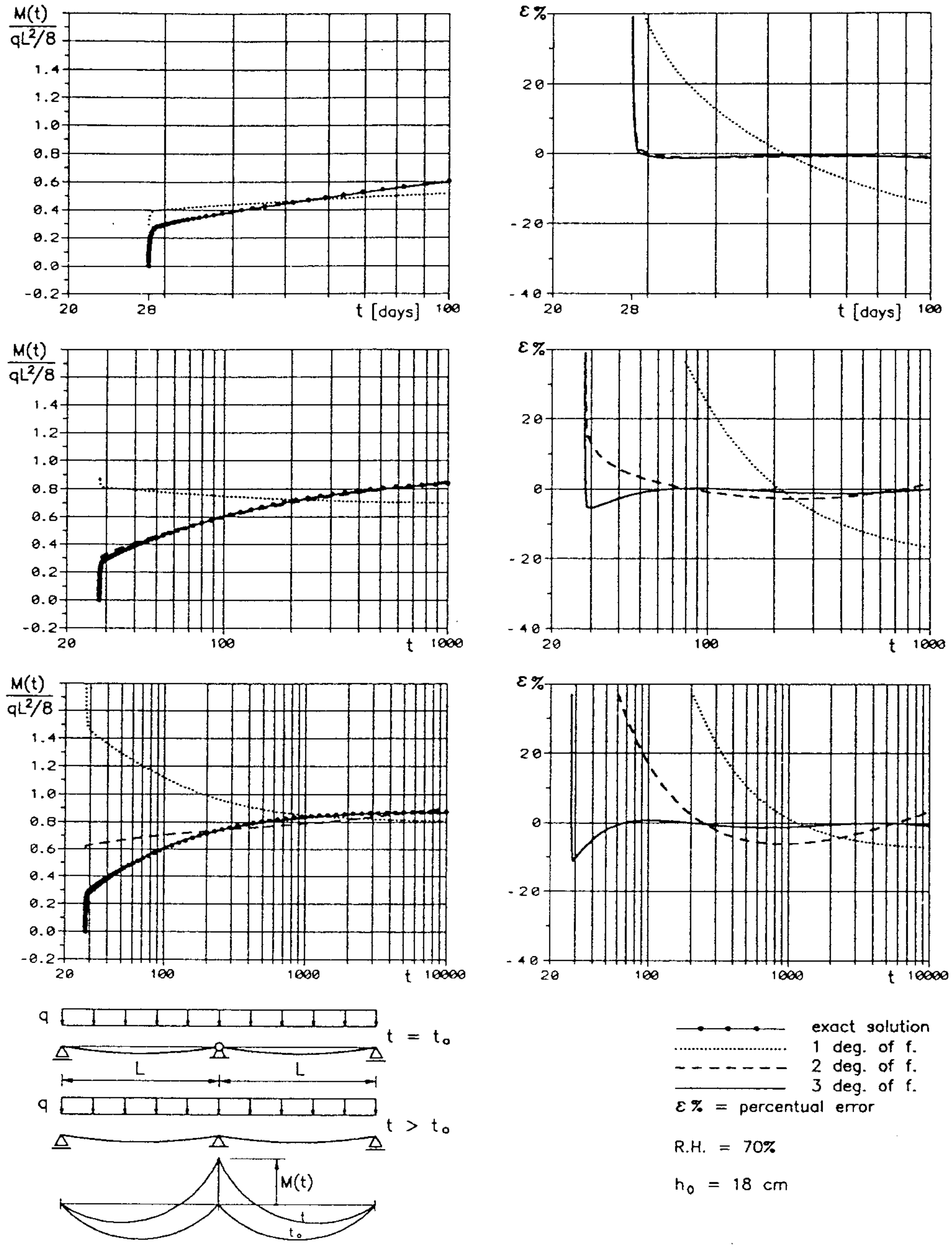


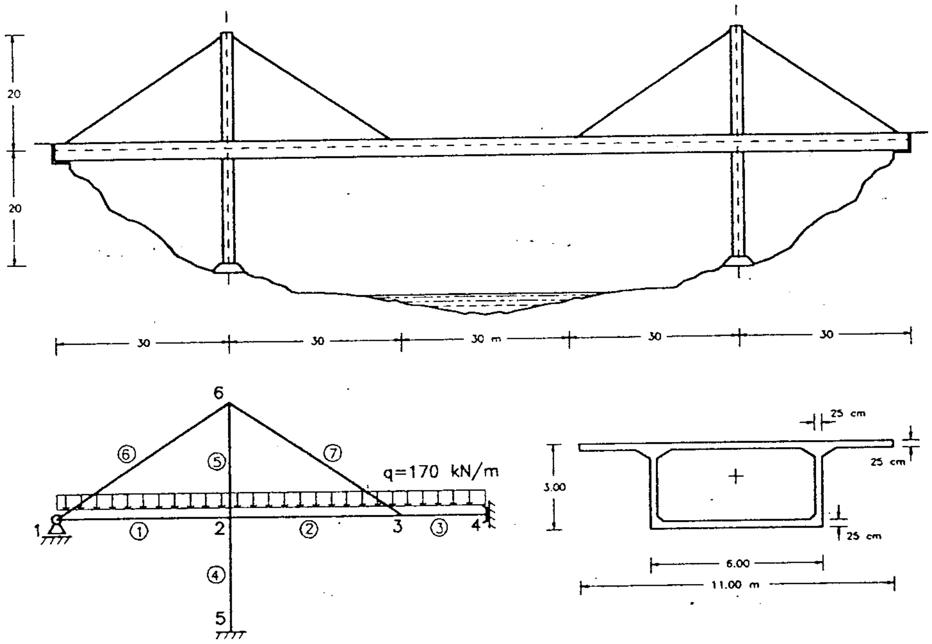
Figure 9. Example c: Beam subjected to a modification of restraint—CEB'78

error has no meaning in this case for times near $t_0 = 28$ d since the exact value tends to zero and therefore the percentage error tends to infinity.

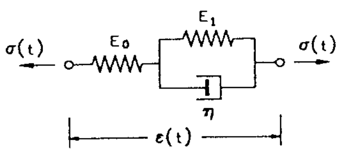
The non-dimensional diagrams of the bending moment show that the solution is satisfactory within the entire range, adopting three degrees of freedom.

Example d. The structure (Figure 10) is typical of a cable stayed bridge. The girder and the piers are made of concrete with geometrical and rheological properties constant along the axis; stays are made of steel. The uniform load q is applied on the girder at time t_0 and remains constant in time.

The structure is discretized with seven elements having the characteristics of Figure 10. The reference solution is obtained with the program Abaqus.[18] In Figures 11 and 12 the diagram of



KELVIN–VOIGT



RHEOLOGICAL PROPERTIES

ELEM.	E_0 kN/m²	E_1 kN/m²	η gg*kN/m²
1	3.80e7	1.90e7	2.70e9
2	3.80e7	1.90e7	2.70e9
3	3.80e7	1.90e7	2.70e9
4	3.00e7	1.50e7	2.70e9
5	3.00e7	1.50e7	2.70e9
6	steel E = 2.10e8		
7	steel E = 2.10e8		

GEOMETRICAL PROPERTIES

ELEM.	AREA m²	J m⁴
1	5.50	8.17
2	5.50	8.17
3	5.50	8.17
4	5.00	2.86
5	2.00	0.67
6	0.04	–
7	0.04	–

CEB '78

Relative Humidity	R.H. = 70 %
Effective Thickness	$h_0 = 3\ A_c/u$
Age of Concrete	t_0

RHEOLOGICAL PROPERTIES

ELEM.	E_{c28} kN/m²	h_0 cm	t_0 days
1	3.80e7	45.8	28
2	3.80e7	45.8	28
3	3.80e7	45.8	28
4	3.00e7	66.9	28
5	3.00e7	100	28
6	steel E = 2.10e8		
7	steel E = 2.10e8		

Figure 10. Example d: Cable-stayed bridge

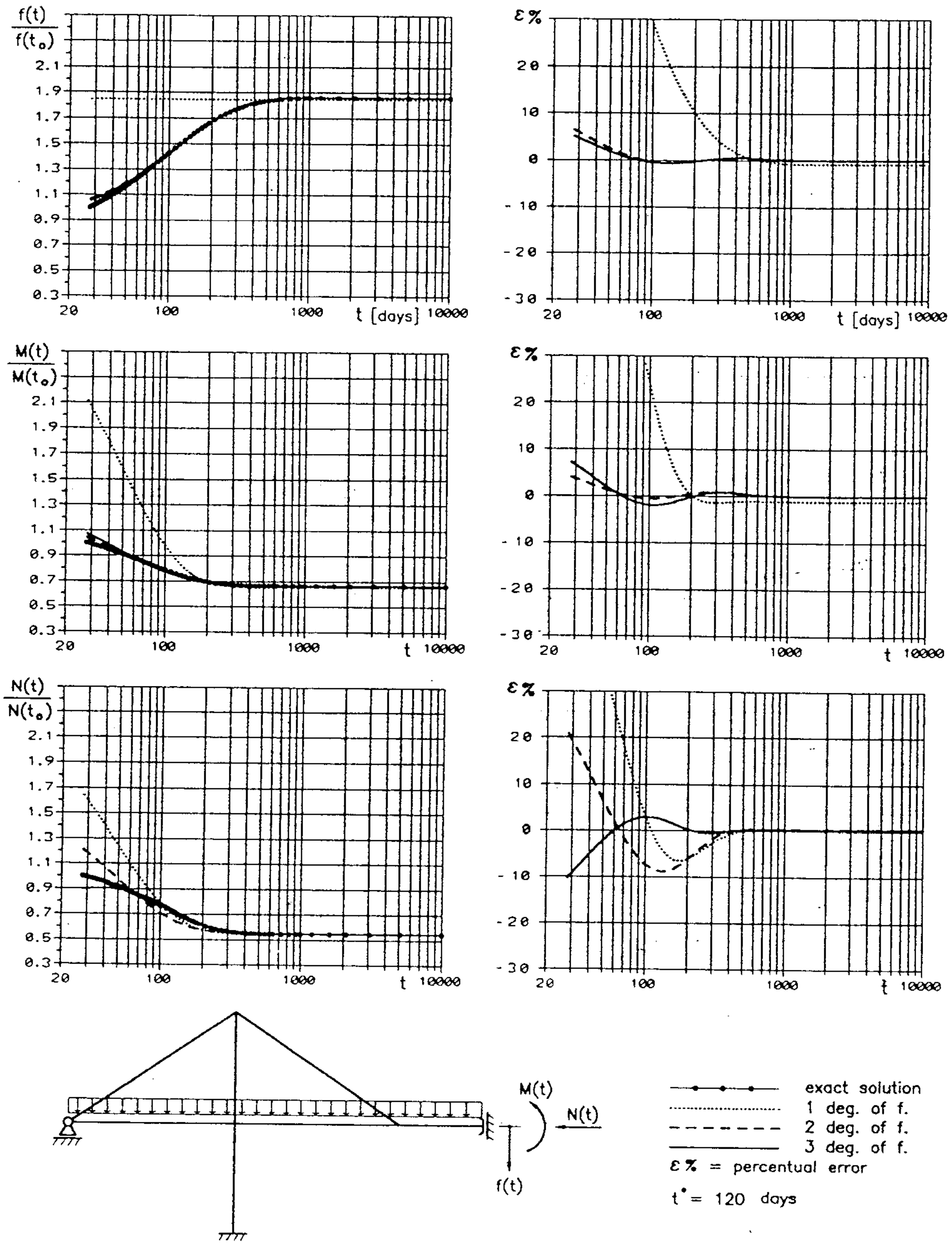


Figure 11. Example d: Cable-stayed bridge—Kelvin–Voigt

the ratios between values at time t and initial elastic values at time t_0 for the deflection, for the bending moment and for the axial force at midspan are plotted.

Figure 11 refers to the Kelvin–Voigt rheological model and Figure 12 to the CEB'78 one.

In Figure 13 the importance of the choice of the parameter t^* for the shape functions of the Kelvin model is emphasized, particularly when few degrees of freedom in time are used.

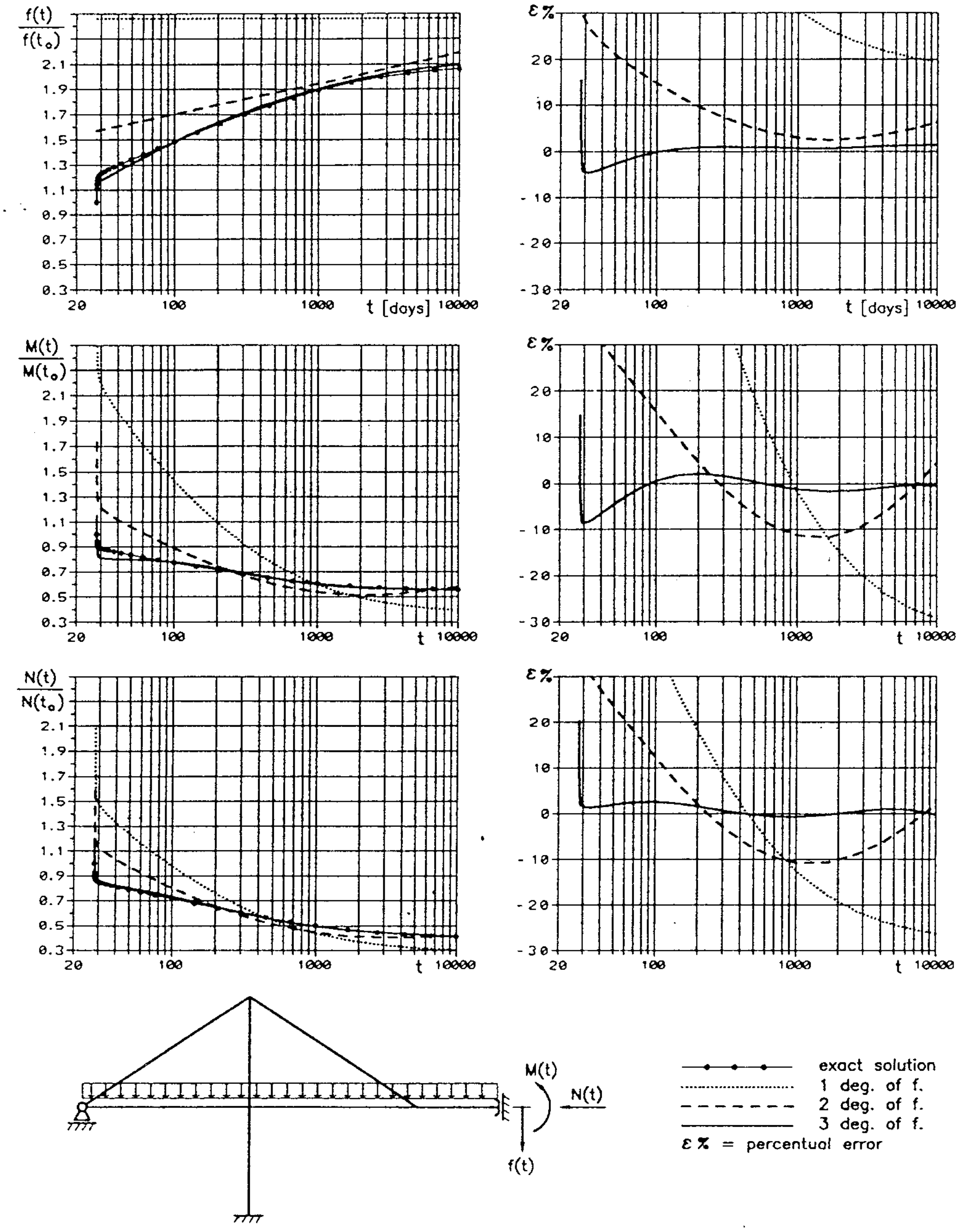


Figure 12. Example d: Cable-stayed bridge—CEB'78

All obtained results are satisfactory for displacements (errors are less than 5 per cent with three degrees of freedom) as well as for internal forces (errors are less than 10 per cent with three degrees of freedom).

The conditioning indexes of the coefficient matrices obtained both with the proposed method and with the least square method are compared, using the same spatial and time discretization. The results are summarized in Table I.

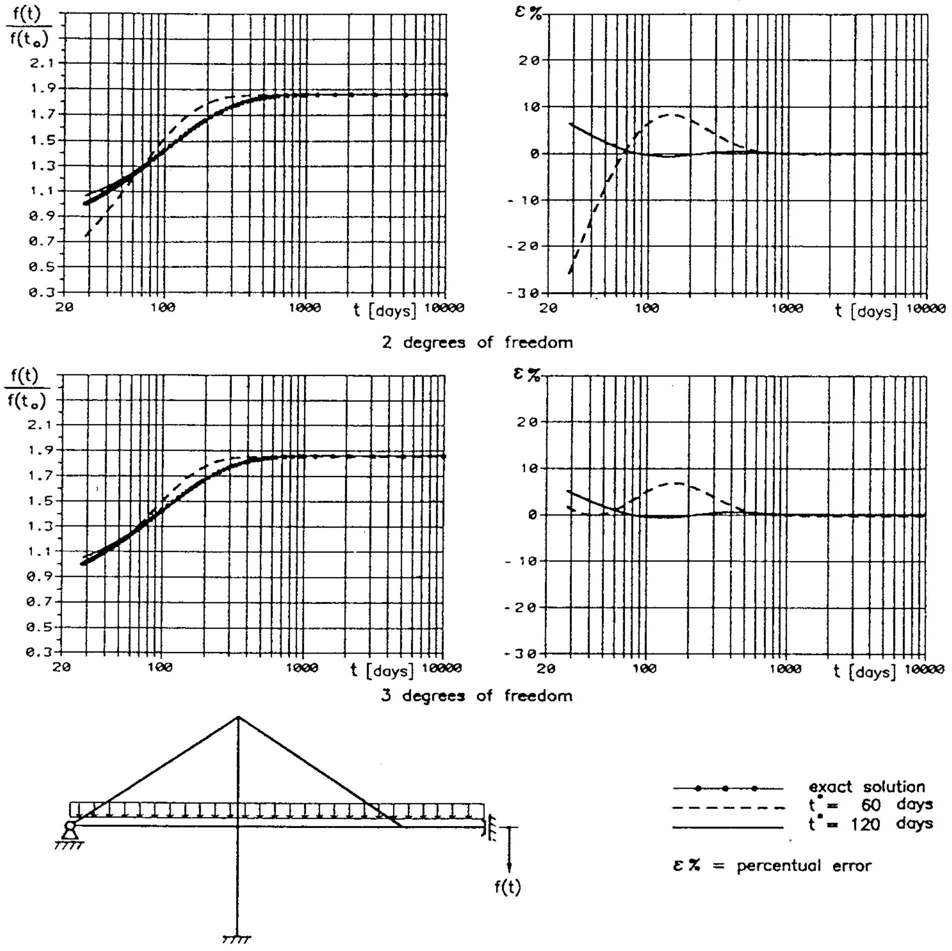


Figure 13. Example d: Cable-stayed bridge—Kelvin–Voigt: weight of parameter t^*

Table I. Conditioning indexes of the coefficient matrices

T(days)	No. of time degrees of freedom	Conditioning index[20] (COND)			
		Least square		Extended functional	
		Kelvin–Voigt	CEB'78	Kelvin–Voigt	CEB'78
28–100	1	$1{\cdot}555 \times 10^{7}$	$1{\cdot}282 \times 10^{7}$	$3{\cdot}635 \times 10^{3}$	$3{\cdot}079 \times 10^{3}$
	2	$7{\cdot}389 \times 10^{8}$	$6{\cdot}852 \times 10^{9}$	$2{\cdot}118 \times 10^{5}$	$6{\cdot}056 \times 10^{6}$
	3	$2{\cdot}705 \times 10^{9}$	$3{\cdot}205 \times 10^{11}$	$2{\cdot}328 \times 10^{7}$	$4{\cdot}108 \times 10^{10}$
28–1000	1	$5{\cdot}944 \times 10^{6}$	$4{\cdot}376 \times 10^{6}$	$2{\cdot}248 \times 10^{3}$	$1{\cdot}887 \times 10^{3}$
	2	$2{\cdot}195 \times 10^{8}$	$7{\cdot}291 \times 10^{10}$	$1{\cdot}051 \times 10^{5}$	$2{\cdot}502 \times 10^{6}$
	3	$1{\cdot}715 \times 10^{10}$	$2{\cdot}261 \times 10^{11}$	$4{\cdot}932 \times 10^{6}$	$4{\cdot}360 \times 10^{9}$
28–10 000	1	$8{\cdot}891 \times 10^{6}$	$2{\cdot}304 \times 10^{6}$	$2{\cdot}064 \times 10^{3}$	$1{\cdot}313 \times 10^{3}$
	2	$2{\cdot}695 \times 10^{9}$	$1{\cdot}744 \times 10^{10}$	$1{\cdot}001 \times 10^{6}$	$5{\cdot}913 \times 10^{6}$
	3	$5{\cdot}796 \times 10^{12}$	$2{\cdot}952 \times 10^{12}$	$8{\cdot}667 \times 10^{7}$	$1{\cdot}206 \times 10^{10}$

As a reference, the conditioning index for the elastic stiffness matrix, obtained with the instantaneous Young modulus at time t_0, is $4{\cdot}8180 \times 10^3$. All indexes are calculated with the program Matlab.[20]

From Table I it is evident that the matrices obtained through the extended functional are better conditioned than those obtained with the least square method. This fact is very important numerically mostly in view of possible applications in the non-linear viscoelastic field.

6. CONCLUSIONS AND REMARKS

In the present paper the following has been presented:

1. an extremal formulation of the linear viscoelastic problem with general viscous kernel in terms of displacements;
2. a corresponding discrete formulation in space and time for framed structures (using finite element technique in space and Ritz technique in time);
3. numerical examples with reference to hereditary Kelvin–Voigt or mixed CEB'78 material model.

It is worth making the following remarks:

(a) In contrast to all the known formulations in viscoelasticity, the here-presented formulation is also valid for ageing materials (like concrete); it can also be easily extended to non-linear viscoelasticity.

(b) As already emphasized in Reference 11, the 'extended functional' represents an energy. This may both help in the comprehension of the method and serve as a tool for the construction of the extended functional for other structural typologies. The easily derivable dual extended functional in terms of stress is here omitted due to its less-frequent numerical use.

(c) From the given examples it appears that the conditioning index of the coefficient matrix of the linear system (for one time degree of freedom) has the same order as that of the corresponding elastic stiffness matrix. This result has been obtained assuming, for the auxiliary problem, the instantaneous relaxation function $R_{ijhk}(\mathbf{x};\, t_0, t_0)$. Instead it is possible, for further improvement of the conditioning matrix index, to choose instantaneous relaxation functions $R_{ijhk}(\mathbf{x};\, \bar{t}, \bar{t})$ with $\bar{t} \in \mathrm{T}$ but $\neq t_0$ (for example $\bar{t} = t_0 + (t_1 - t_0)/2$).

Good results of the method depends on the choice made for the Tonti's 'integrating operator' $\mathbb{K}$.[12] Unwise choices may lead to matrices with far worse conditioning indexes. This is emphasized here for the choice of $\mathbb{K} = \mathbb{I}$ (identity operator, to which the least square method corresponds). In any case the discretization leads to a linear equation system with a symmetric and positive-definite coefficient matrix.

(d) In all the examples given, the reliability and the accuracy of the method is evident. With three time degrees of freedom, results are affected almost everywhere by errors less than 5 per cent. The structural response, different from step-by-step methods, is given immediately on the whole temporal range without the need of time integrations for the step-by-step solution to proceed.

(e) It is possible to apply the present method on subintervals of the original integration time range T, therefore using a step-by-step type procedure. This implies a reduced number of time degrees of freedom over each subinterval and the construction of smaller matrices. Results in this direction will be presented in a subsequent paper (Reference 19) of which an example is presented in advance in Figure 3.

(f) The proposed method requires the inversion of the elastic stiffness matrix in order to construct the solving linear system. This may involve numerical problems when the system has many degrees of freedom. It is worth noting that the matrices obtained are generally full. These drawbacks may however be overcome through a method proposed by Ortiz.[21] Studies by the authors on this matter are at the moment in progress and will be presented in a subsequent paper (Reference 22).

(g) In the context of the boundary integral equations (BIE) method, recent contributions for elastic continua[23,24] permitted to attain, even under a BIE approach, a variational formulation in space and time (using the convolution bilinear form in time) of linear hereditary viscoelasticity and a mini-max formulation (using the Reiss bilinear form in time).[25] It is also possible to extend the present method to the boundary integral equations in linear viscoelasticity with a general viscous kernel (not only a hereditary kernel) by obtaining an extension of the mini-max formulation shown in Reference 25. A parallel work[26] points out the results in this direction.

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REFERENCES

1. M. A. Biot, 'Variational and Lagrangian methods in viscoelasticity' in R. Grammel (ed.), *Deformation and Flow of Solids, Proc. IUTAM, Kolloquium Madrid*, 26–30 September 1955, Springer, Berlin, 1956, pp. 251–263.
2. E. T. Onat, 'On a variational principle in linear viscoelasticity', *Journal de Mécanique*, **I**(2), 134–140 (1962).
3. M. E. Gurtin, 'Variational principles in the linear theory of viscoelasticity', *Arch. Rat. Mech. Anal.*, **13**, 179–191 (1963).
4. J. Brilla, 'Convolutional variational principles and methods in linear viscoelasticity', in *Variational Methods in Engineering, Proc. Int. Conf.*, Department of Civil Engineering University of Southampton, Southampton, 25 September 1972.
5. R. M. Christensen, 'Variational and minimum theorems for the linear theory of viscoelasticity', *J. Appl. Math. Phys., Zamp.* **19**, 233–243 (1968).
6. P. Rafalski, 'The orthogonal projection method II. Linear viscoelastic problem', *Bull. Acad. Polon. Sci., Ser. Sci. Techn.*, **17**, 167, (1969).
7. P. Rafalski, 'A method of solution of the linear boundary problems of heat conduction, thermoelasticity and viscoelasticity', *Proc. 1st Int. Conf. on Struct. Mechanics in Reactor Technology*, Berlin, 1972, Vol. 6, No. 7/2, p. 489.
8. P. Rafalski, 'On extremum principles for initial-boundary value problems', *Lett. Appl. Eng. Sci.*, **17**, 793–795 (1979).
9. R. Reiss and E. J. Haug, 'Extremum principles for linear initial value problems of mathematical physics', *Int. J. Eng. Sci.*, **16**, 231–251 (1978).
10. C. Huet, 'Minimum Theorems for viscoelasticity', *Eur. J. Mech., A/Solids*, **11**(5), 653–684 (1992).
11. A. Carini, 'Colonnetti's minimum principle extension to generally nonlinear materials' (to appear).
12. E. Tonti, 'Variational formulation for every nonlinear problem', *Int. J. Eng. Sci.*, **22**(11/12), 1343–1371 (1984).
13. J. Lubliner and J. L. Sackman, 'On uniqueness in general linear viscoelasticity', *Quart. Appl. Math.*, **XXV**(2), 129–138 (1967).
14. R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer, New York, 1984.
15. M. E. Gurtin and E. Sternberg, 'On the linear theory of viscoelasticity', *Arch. Rat. Mech. Anal.*, **11**, 291–356 (1962).
16. R. M. Christensen, *Theory of Viscoelasticity: an Introduction*, Academic Press, New York, 2nd edn, 1982.
17. 'CEB Manual on structural effects of time-dependent behaviour of concrete', *Bulletin d'Information* N. 142/142 bis, Mars 1984.
18. *ABAQUS V. 4.9 User's and Theory Manuals*, Hibbit Karlsson & Sorensen Inc., Rhode Island, 1991.
19. A. Carini, P. Gelfi and E. Marchina, 'An energetic formulation for the linear viscoelastic problem. Part II: step-by-step time integration approach' (in progress).
20. J. Little and C. Moler, *MATLAB User's Guide*, Portola Valley, CA, 1985.
21. M. Ortiz, 'A variational formulation for convection–diffusion problems', *Int. J. Eng. Sci.*, **23**(7), 717–731 (1985).
22. A. Carini, P. Gelfi and E. Marchina, 'An energetic formulation for the linear viscoelastic problem. Part III: spatial element-by-element integration approach' (in progress).

23. S. Sirtori, 'General stress analysis by means of integral equation and boundary elements', *Meccanica*, **14**, 210–218 (1979).
24. C. Polizzotto, 'An energy approach to the boundary element method. Part I: Elastic solids', *Comput. Methods Appl. Mech. Eng.*, **60**, 175–194 (1987).
25. A. Carini, M. Diligenti and G. Maier, 'Boundary integral equation analysis in linear viscoelasticity: variational and saddle point formulations' *Comp. Mech.*, **8**, 87–98 (1991).
26. A. Carini and M. Diligenti, 'An energetic formulation for the boundary integral equation in linear viscoelasticity', (in progress).